MTH 265. Week 3 Wednesday. Lecture Notes.
Ratio Tests and Root Tests.

Theorem 1. The Ratio Test.

Let $\sum_{n=n_0}^{\infty} a_n$ be a series and let $L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$;

① If $L < 1$, then the series $\sum_{n=n_0}^{\infty} a_n$ is absolutely convergent.

② If $L = 1$, the Ratio Test is inconclusive.

③ If $L > 1$ or $L = \infty$, then the series $\sum_{n=n_0}^{\infty} a_n$ is divergent.

Proof Sketch. The ratio test measures how the terms of the series change as n increases.
If $L < 1$, then the series $\sum |a_n|$ can be bounded above with a geometric series $\sum |a_{n_0}| r^n$ for some $r \in (L, 1)$.
By construction, $r \in (0,1)$ and $\sum |a_n|$ converges by the Comparison Test.
If $L > 1$, then the terms of $\sum |a_n|$ are, eventually for sufficiently high n, strictly increasing.
Then, $\lim_{n\to\infty} |a_n| \neq 0$ and $\sum a_n$ diverges by the Divergence Test.

Example 1.1. Let $a_n = ne^{-n}$; Determine if $\sum_{n=1}^{\infty} ne^{-n}$ converges or diverges.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)e^{-(n+1)}}{ne^{-n}} = \lim_{n\to\infty} \frac{(n+1)}{n} \cdot \frac{1}{e} = (1)\left(\frac{1}{e}\right) < 1. \quad \therefore \sum_{n=1}^{\infty} ne^{-n} \text{ converges by the Ratio Test.}$$

Example 1.2. Let $a_n = \frac{1}{n!}$; Determine if $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges or diverges.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{n!}{(n+1)n!} = \lim_{n\to\infty} \frac{1}{n+1} = 0; \quad \therefore \sum_{n=1}^{\infty} \frac{1}{n!} \text{ Converges by the Ratio Test.}$$

Example 1.3. Determine if $\sum_{n=1}^{\infty} \frac{(-9)^n}{n(10^{n+1})}$ converges;

Let $a_n = \frac{(-9)^n}{n(10^{n+1})}$; $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n(10^{n+1})}{9^n} \cdot \frac{9^{n+1}}{(n+1)10^{n+2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)\left(\frac{9}{10}\right) = \frac{9}{10} < 1.$

$\therefore \sum_{n=1}^{\infty} \frac{(-9)^n}{n(10^{n+1})}$ absolutely converges by the Ratio Test.

Example 1.4. Determine if $\sum_{n=1}^{\infty} n^{-2} 3^n$ diverges or converges.

Let $a_n = \frac{3^n}{n^2}$; $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n^2}{3^n} \cdot \frac{3^{n+1}}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2}(3) = 3 > 1$; $\sum_{n=1}^{\infty} \frac{3^n}{n^2}$ diverges by the Ratio Test.

Non-example 1.5. Determine if $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the Ratio Test.

Let $a_n = \frac{1}{n^2}$; $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \right| = 1$; The Ratio Test is inconclusive;

Example 1.6. Determine if $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$ converges;

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{((n+1)!)^2}{(2n+2)!}\cdot\frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \frac{(n+1)^2\cancel{(n!)^2}}{(2n+2)(2n+1)\cancel{(2n)!}}\cdot\frac{\cancel{(2n)!}}{\cancel{(n!)^2}} = \lim_{n\to\infty}\frac{(n+1)^2}{(2n+2)(2n+1)}$$

$= \frac{1}{4} < 1$. $\sum_{n=1}^{\infty}\frac{(n!)^2}{(2n)!}$ converges by the Ratio Test.

Theorem 2. The Root Test.

Let $\sum_{n=n_0}^{\infty} a_n$ be a series and let $L = \lim_{n\to\infty}\sqrt[n]{|a_n|}$;

① If $L<1$: the series $\sum_{n=n_0}^{\infty} a_n$ is absolutely convergent.

② If $L=1$: the Ratio Test is inconclusive;

③ If $L>1$ or $L=\infty$: the series $\sum_{n=n_0}^{\infty} a_n$ is divergent.

Proof Sketch. The proof for the Root Test is very similar to that of the Ratio Test.
If $L<1$, we can choose $r\in(L,1)$ such that $|a_n|\le r^n$ for sufficiently high n;
Then, by the Comparison Test, $\Sigma|a_n|$ converges.
If $L>1$, then $|a_{n+1}|>|a_n|$ for sufficiently high n. Then, $\Sigma|a_n|$ diverges and Σa_n diverges both by the Divergence Test.

Example 2.1. Determine if $\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^n$ converges;

Let $a_n = \left(\frac{2n+3}{3n+2}\right)^n$; $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left[\left(\frac{2n+3}{3n+2}\right)^n\right]^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{2n+3}{3n+2}\right) = \frac{2}{3} < 1$.

$\therefore \sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)$ is absolutely conv. by the Root Test.

Example 2.2. Determine if $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(\ln(n))^n}$ converges. Let $a_n = \frac{(-1)^{n-1}}{(\ln(n))^n}$;

$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\frac{1}{(\ln(n))^n}} = \lim_{n\to\infty}\frac{1}{\ln(n)} = 0$; $\therefore \sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{(\ln(n))^n}$ absolutely converges by the Root Test.

Example 2.3. Determine if $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$ is convergent.

$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ by definition. $\therefore \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$ diverges by the Root Test;

Example 2.4. Determine if $\sum_{n=1}^{\infty}\frac{2^n}{n^n}$ is convergent. $\lim_{n\to\infty}\sqrt[n]{\frac{2^n}{n^n}} = \lim_{n\to\infty}\frac{2}{n} = 0 < 1$. $\sum_{n=1}^{\infty}\frac{2^n}{n^n}$ conv. by the Root Test.